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EFFECT OF SCRAP AND REWORK COST ON OPTIMUM PRODUCTION PROCESS MEAN VALUE

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ABSTRACT - *The costs due to producing a product below the lower specification and above the upper specification limit are not equal in the production process. Due to these different costs, the total cost incurred will greatly depend on the process mean value. This study shows how the optimum value of production process mean can be obtained when the quality characteristic follows normal and non-normal distributions. In this study, optimization models for determining the optimum values of production process mean under different cost criteria are presented. A numerical example and sensitivity analysis are performed for each quality characteristic distribution.*

KEYWORDS: *Process mean, Rework cost, Scrap cost.*

1. INTRODUCTION

Modern industries are constantly striving to reduce costs in their process. Process cost reduction entails a rational decision making in relation to the process parameters using quantitative modeling. One of the most important decision making problems encountered in a wide variety of industrial process is determination of process mean.

Carlsson [1] derived the optimal process mean, assuming that a net income function is a linear function of a quality characteristic. Golhar [2] determined the best mean contents for a canning process. He assumes that, under filled cans are not sold but reprocessed and sold at regular price. Golhar and Pollock [3] determined the optimum process mean and upper limit for a canning process. This model was an extension of Golhar's model [2] in which both the upper limit and mean are taken as control variables. Golhar and Pollock [4] studied the effect of reduction in filling process variance on the saving excess cost. The linear profit function of quality characteristic has been applied in their analysis for determining the optimum process mean. Dodson [5] derived the optimum process target value for a production process under different cost criteria. He assumes that, the products that fall above the upper limit are scrapped. Madduri et al [6] presented the optimal process mean under different cost environments for a process modeled by a normal distribution. Misiorek and Barnett [7] presented the modified Golhar and Pollock model [3] with extra cost parameters. They considered the normal distribution of the quality characteristic in their analysis. Lee et al [8,9] studied the economic selection of the process mean. The normal distribution of the quality characteristic has been considered in their analysis. Li [10] studied the economic selection of process mean. The quadratic and linear quality loss function has been applied in his study for determining the optimum process mean value. Chung [11] presented a method for determining the optimum process mean in order to minimize the expected total loss of a product. This method takes into account an indirect quality characteristic with the quadratic quality loss of a product within specifications. Chung [12] presented the modified Misiorek and Barnett's model [7] with beta distribution for determining the optimum process mean.

The majority of the previous studies assume that, the quality characteristic of the production process follows a normal distribution. But the quality characteristic distribution is more flexible in shape such as lognormal distribution (skewed distribution). In this study, an optimization model for determining the optimum production process mean value when the quality characteristic follows normal and non-normal distribution has been presented. A numerical example and sensitivity analysis are performed for each quality characteristic distribution.

2. THEORETICAL ANALYSIS

Two types of quality costs are considered in this study. First, when a product falls below the lower specification limit, the product is rejected and a scrap cost (C_R) is incurred. Second, when a product falls above the upper specification limit, the product is reworked and a rework cost (C_E) is incurred. Assume that, all products are subject to inspection (100 % inspection) and no inspection error is allowed. Optimization models under two cases of quality characteristic distribution will be discussed in the following sections.

2.1 Case 1 : Quality Characteristic Is Normally Distributed

Assume that the quality characteristic Y is normally distributed with a mean μ and a standard deviation σ. Let L and U denote the lower and upper specification limits, respectively.

2.1.1 Minimizing expected total cost model

The objective is to minimize the total cost considering scrap and rework costs. Under these situations, the optimization model is to

$$\text{Minimize } E(T_C) = E(C_R) + E(C_E) \tag{1}$$

Subject to $L \leq \mu \leq U$

Where $E(T_C)$ is the expected total cost per product unit, $E(C_R)$ is the expected scrap cost due to rejected products which fall below L, $E(C_E)$ is the expected rework cost due to products which fall above U. The constraint serves to ensure that the process target falls between the specification limits. The objective function in equation 1 becomes

$$E(T_C) = C_E\, P(Y \geq U) + C_R\, P(Y \leq L)$$
$$= C_E - C_E\, \Phi\left[\frac{U-\mu}{\sigma}\right] + C_R\, \Phi\left[\frac{L-\mu}{\sigma}\right] \quad (2)$$

Where $\Phi(.)$ represent the cumulative normal density function. The first derivative of equation 2 with respect to μ, can be obtained as

$$\frac{d\,(E[T_C])}{d\,\mu} = \frac{-C_R}{\sigma}\,\phi\left(\frac{L-\mu}{\sigma}\right) + \frac{C_E}{\sigma}\,\phi\left(\frac{U-\mu}{\sigma}\right) \quad (3)$$

Where $\phi(.)$ denotes the standard normal density function. Setting equation 3 to zero,

$$\frac{1}{\sigma}\left[C_E\,\phi\left(\frac{U-\mu}{\sigma}\right) - C_R\,\phi\left(\frac{L-\mu}{\sigma}\right)\right] = 0 \quad (4)$$

Note that $\left(1/\sigma\right)$ is not equal to zero from a practical view point and hence

$$C_E\,\phi\left(\frac{U-\mu}{\sigma}\right) = C_R\,\phi\left(\frac{L-\mu}{\sigma}\right) \quad (5)$$

Equation 5 reduces to

$$\frac{C_R}{C_E} = \exp\left(\frac{1}{2}\left[\left(\frac{L-\mu}{\sigma}\right)^2 - \left(\frac{U-\mu}{\sigma}\right)^2\right]\right) \quad (6)$$

Taking a logarithmic transformation, equation 6 becomes

$$\ln\left(\frac{C_R}{C_E}\right) = \frac{1}{2}\left(\frac{L^2-U^2}{\sigma^2}\right) + \left(\frac{U\,\mu - L\,\mu}{\sigma^2}\right) \quad (7)$$

$$\mu\left(\frac{U-L}{\sigma^2}\right) = \ln\left(\frac{C_R}{C_E}\right) - \frac{1}{2}\left(\frac{L^2-U^2}{\sigma^2}\right) \quad (8)$$

After some algebra, equation 8 becomes

$$\mu^* = \frac{\sigma^2\left[\ln\left(\frac{C_R}{C_E}\right)\right]}{U-L} + \frac{U+L}{2} \quad (9)$$

The second derivative of equation 2 is always greater than zero, which indicates that equation 2 is a convex function. Hence equation 9 provides the global minimum (optimum) for the process mean.

2.1.2 Maximizing expected profit model

Suppose a product that falls in between two limits can be sold at price P in the market. The model to maximize the profit is then to

$$\text{Maximize } E\left[N_P\right] = E\left[C_P\right] - E\left[C_R\right] - \left[C_E\right] \tag{10}$$

Subject to $L \le \mu \le U$

Where $E\left[N_P\right]$ is the expected net (total) profit, $E\left[C_P\right]$ is the expected revenue per unit. The objective function in equation 10 can be rewritten as

$$E\left[N_P\right] = P\left[\phi\left(\frac{U-\mu}{\sigma}\right) - \phi\left(\frac{L-\mu}{\sigma}\right)\right] - C_R\,\phi\left(\frac{L-\mu}{\sigma}\right) - C_E + C_E\,\phi\left(\frac{U-\mu}{\sigma}\right) \tag{11}$$

Taking the first derivative of equation 11 with respect to μ and setting it equal to zero results in

$$\frac{\phi\left(\dfrac{U-\mu}{\sigma}\right)}{\phi\left(\dfrac{L-\mu}{\sigma}\right)} = \left(\frac{P+C_R}{P+C_E}\right) \tag{12}$$

After some algebra, equation 12 becomes

$$\exp\left(\frac{1}{2\,\sigma^2}\left(L^2 - U^2 + 2\,\mu\left(U-L\right)\right)\right) = \left(\frac{P+C_R}{P+C_E}\right) \tag{13}$$

After applying a logarithmic transformation, equation 13 reduces to

$$\frac{\mu\left(U-L\right)}{\sigma^2} = \ln\left(\frac{P+C_R}{P+C_E}\right) + \left(\frac{U^2-L^2}{2\,\sigma^2}\right) \tag{14}$$

Hence,

$$\mu^* = \frac{\sigma^2\left[\ln\left(\dfrac{P+C_R}{P+C_E}\right)\right]}{U-L} + \left(\frac{U+L}{2}\right) \tag{15}$$

The objective function in equation 10 is concave and therefore equation 15 provides the global maximum.

2.2 Case 2 : Quality Characteristic Is Non -Normally Distributed

Assumed that the quality characteristic Y follows a lognormal distribution and its probability density function is given by

$$f(y) = \frac{1}{y\,\sigma\,\sqrt{2\pi}}\, e^{\frac{-1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2} \tag{16}$$

Where μ and σ are the mean and standard deviation of the normal distribution associated with the lognormal variable y.

The expected total cost $E(T_C)$ per product unit (equation 1)

$$E(T_C) = E(C_R) + E(C_E)$$

In terms of the probability of the products falling outside the limits, the expected total cost equation can be written as

$$E(T_C) = C_E\, P(Y \geq U) + C_R\, P(Y \leq L)$$

By using the definition of an expected value of a random variable, the expected total cost can be given by

$$E(T_C) = C_R \int_0^L f(y)\,dy + C_E \int_U^0 f(y)\,dy$$

2.2.1 Minimizing expected total cost model

The objective is to minimize the expected total cost. Hence the optimization model is to

$$\text{Minimize } E(T_C) = C_E\left[1-\Phi\left(\frac{\ln U-\mu}{\sigma}\right)\right] + C_R\,\Phi\left(\frac{\ln L-L}{\sigma}\right) \tag{17}$$

Subject to $L \leq \mu \leq U$

To determine the optimum process mean that would minimize the expected total cost, equation 17 is differentiated with respect to μ and the first derivative is given by

$$\frac{d\,(E(T_C))}{d\mu} = \frac{-C_R}{\sigma}\,\phi\left(\frac{\ln L-\mu}{\sigma}\right) + \frac{C_E}{\sigma}\,\phi\left(\frac{\ln U-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma}\left(C_E\,\phi\left(\frac{\ln U-\mu}{\sigma}\right) - C_R\,\phi\left(\frac{\ln L-\mu}{\sigma}\right)\right) \tag{18}$$

Since σ is always positive, $\left(1/\sigma\right)$ can never be equal to zero. Therefore, equation 18 can then be written as

$$C_E\ \phi\left(\frac{\ln U-\mu}{\sigma}\right)=C_R\ \phi\left(\frac{\ln L-\mu}{\sigma}\right) \tag{19}$$

The expression for probability density function given in equation 16 can be used to expand equation 19 as

$$\frac{C_R}{C_E}=\exp\left(\frac{1}{2}\left[\left(\frac{\ln L-\mu}{\sigma}\right)^2-\left(\frac{\ln U-\mu}{\sigma}\right)^2\right]\right) \tag{20}$$

Taking the logarithm of both sides, equation 20 becomes

$$\ln\left(\frac{C_R}{C_E}\right)=\frac{1}{2}\left(\frac{(\ln L)^2-(\ln U)^2}{\sigma^2}\right)+\left(\frac{(\ln U)\times\mu-(\ln L)\times\mu}{\sigma^2}\right) \tag{21}$$

Simplifying equation 21, optimal process mean can be given by

$$\mu^*=\frac{\sigma^2\left[\ln\left(\frac{C_R}{C_E}\right)\right]}{\ln\left(\frac{U}{L}\right)}+\frac{\ln(U\times L)}{2} \tag{22}$$

The optimal mean obtained in equation 22 pertains to the distribution of $\ln(y)$. The optimal process mean for the lognormal distribution of y can then be given by $e^{\mu^*+\frac{\sigma^2}{2}}$. The second derivative of equation (17) is always greater than zero. Hence, equation 17 is a convex function and the optimal solution obtained in equation 22 is a global minimum. Thus, equation 22 gives the value of process mean for the minimum process cost.

2.2.2 Maximizing expected profit model

In this model, the maximization of expected profit is the optimization criterion. If a product that falls within the screening limits can be sold at a price P, then expected profit is given by (see equation 10)

$$E[N_P]=E[C_P]-E[C_R]-E[C_E]$$

$$E[C_P]=\ P\,(p\,(Y\le U)-p\,(Y\le L))$$

$$=P\left[\Phi\left(\frac{\ln U-\mu}{\sigma}\right)-\Phi\left(\frac{\ln L-\mu}{\sigma}\right)\right]$$

Therefore, the expected profit can be given by

$$E\left[N_P\right] = P\left[\Phi\left(\frac{\ln U-\mu}{\sigma}\right)-\Phi\left(\frac{\ln L-\mu}{\sigma}\right)\right]$$
$$-C_E\left[1-\Phi\left(\frac{\ln U-\mu}{\sigma}\right)\right]-C_R\Phi\left(\frac{\ln L-\mu}{\sigma}\right) \tag{23}$$

By differentiating $E\left[N_P\right]$ with respect to μ and the first derivative is given by

$$\frac{d\left(E\left[N_P\right]\right)}{d\mu}=\frac{-1}{\sigma}\left(\begin{array}{l} P\left[\phi\left(\frac{\ln U-\mu}{\sigma}\right)-\phi\left(\frac{\ln L-\mu}{\sigma}\right)\right] \\ +C_E\,\phi\left(\frac{\ln U-\mu}{\sigma}\right)-C_R\,\phi\left(\frac{\ln L-\mu}{\sigma}\right)\end{array}\right) \tag{24}$$

To obtain the optimal solution, $\frac{d\left(E\left[N_P\right]\right)}{d\mu}$ is equal to zero. Since $\left(1/\sigma\right)$ is always greater than zero. Therefore, equation 24 can then be written as

$$P\left[\phi\left(\frac{\ln U-\mu}{\sigma}\right)-\phi\left(\frac{\ln L-\mu}{\sigma}\right)\right]=C_R\,\phi\left(\frac{\ln L-\mu}{\sigma}\right)-C_E\,\phi\left(\frac{\ln U-\mu}{\sigma}\right) \tag{25}$$

Dividing equation 25 by $\phi\left(\frac{\ln U-\mu}{\sigma}\right)$ and rearranging the terms, equation 25 becomes

$$\frac{\phi\left(\frac{\ln U-\mu}{\sigma}\right)}{\phi\left(\frac{\ln L-\mu}{\sigma}\right)}=\left(\frac{P+C_R}{P+C_E}\right) \tag{26}$$

By using expression for probability density function in equation 16, equation 26 can be expanded to

$$\exp\left(\frac{1}{2\,\sigma^2}\left(\left(\ln L\right)^2-\left(\ln U\right)^2+2\mu\left(\ln U\right)-\ln L\right)\right)=\frac{P+C_R}{P+C_E} \tag{27}$$

Considering the logarithmic transformation on both sides, equation 27 can be written as

$$\frac{\mu\left(\ln U-\ln L\right)}{\sigma^2}=\ln\left(\frac{P+C_R}{P+C_E}\right)+\left(\frac{\left(\ln U\right)^2-\left(\ln L\right)^2}{2\sigma^2}\right)$$

Therefore, the optimum process mean can be given by

$$\mu^* = \frac{\sigma^2\left[\ln\left(\frac{P+C_R}{P+C_E}\right)\right]}{\ln\left(\frac{U}{L}\right)} + \frac{\ln\left(U \times L\right)}{2} \qquad (28)$$

The optimum process mean for the lognormal variable y is given by $e^{\mu^* + \frac{\sigma^2}{2}}$. The second derivative of equation 23 is always less than zero. Hence, equation 23 is a convex function and the optimal solution obtained in equation 28 is a global minimum. Thus, equation 28 provides the global maximum. Hence implementing the process mean, using equation 28 the maximum profit will be obtained.

3. NUMERICAL EXAMPLE AND DISCUSSION

In this section, an example is presented and then based on this example, sensitivity analysis is performed to show how the effects of the change in the cost ratio $\left[\frac{C_R}{C_E}\right]$ on the optimum process mean value $\left(\mu^*\right)$. This example is taken from [6], so that the solution can be compared with the solution obtained based on the assumption of normality. Here is some information about this example, the variance of normal distribution for the quality characteristic associated with variable $\sigma^2 = 0.5$. The lower and upper specification limits are $L = 1$ and $U = 7$, respectively. A product which falls in between two specification limits can be sold at price \$ 5 $\left(P = \$5\right)$ in the market. The results of sensitivity analysis by varying the cost ratio value $\left[\frac{C_R}{C_E}\right]$ are shown in **Table 1**.

It can be noticed that, the optimum process mean value varies depending on the ratio between the scrap cost and rework cost. When the quality characteristic follows normal distribution the optimum process mean value is higher than that of the quality characteristic follows non-normal distribution. When the scrap and rework costs are equal $\left(\frac{C_R}{C_E} = 1\right)$, the optimum process mean value using model 1 (minimizing expected total cost) or using model 2 (Maximizing Expected Profit) is equal. But as the scrap cost increases, the optimum process mean value increases for the two models when the quality characteristic follows normal and non-normal distributions. That is, the optimum mean shifts toward to the upper specification limit $\left(U\right)$.

Table 1: Calculated results of optimum process mean value.

Cost Ratio $\left(\frac{C_R}{C_E}\right)$	Optimum Process Mean Value, μ^*			
	Normal Distribution		Non-Normal Distribution	
	Model 1 Minimizing Expected Total Cost	Model 2 Maximizing Expected Profit	Model 1 Minimizing Expected Total Cost	Model 2 Maximizing Expected Profit
0.25	3.882	3.971	2.091	2.697
0.50	3.943	3.966	2.503	2.875
0.75	3.974	3.987	2.784	2.949
1.00	4.000	4.000	2.992	2.992
1.25	4.012	4.010	3.164	3.012
1.50	4.032	4.010	3.312	3.034
1.75	4.054	4.010	3.453	3.049
2.00	4.062	4.010	3.573	3.056
2.25	4.074	4.010	3.684	3.063
2.50	4.082	4.010	3.784	3.070
2.75	4.084	4.010	3.872	3.077
3.00	4.091	4.010	3.963	3.080
3.25	4.101	4.010	4.043	3.084
3.50	4.103	4.010	4.124	3.087
3.75	4.111	4.010	4.194	3.091
4.00	4.122	4.011	4.264	3.093
4.25	4.123	4.011	4.334	3.095
4.50	4.132	4.012	4.394	3.098
4.75	4.133	4.012	4.453	3.099
5.00	4.134	4.012	4.514	3.100

C_E : Rework cost per unit.

C_R : Scrap cost per unit

4. CONCLUSIONS

From the previous discussion the following conclusions can be drawn:

1. The assumption of normality for quality characteristic distribution in the production process may not be always appropriate.
2. By using the presented analytical techniques, the optimum value of production process mean can be determined for normal and non-normal quality characteristic distributions.
3. By varying the cost parameters, such as scrap cost and rework cost the sensitivity analysis revealed the behavior of the optimum production process mean under different distributions of the quality characteristic.
4. The optimum production process mean using maximizing expected profit model is higher than the one using minimizing expected total cost model for each quality characteristic distribution.

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تأثير تكلفة الخردة وإعادة التشغيل على القيمة المتوسطه المثلى لعملية الإنتاج

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فـي معظـم عمليـات التصـنيع تكـون التكـاليف الناتجـة عـن إنتـاج عناصـر ذات مواصفات اقل من الحد المسموح به لا تتساوى مع التكاليف الناتجة عن إنتاج عناصر ذات مواصفات اكبر من الحد المسموح بـه. بسبب هذا الاختلاف تعتمد التكاليف الكلية لعملية الإنتاج اعتمادا كبيرا على القيمة المتوسطه لعملية الإنتاج.

اجريت هذه الدراسة بهدف تحديد القيمة المتوسطه المثلى لعملية الإنتاج في حالة إتباع صفة الجودة التوزيع الطبيعي أو التوزيع الغير طبيعي. تم اشتقاق الصيغة الرياضية و التي بواسطتها أمكن تحديد القيمة المتوسطه المثلى لعملية الإنتاج وذلك عند اختلاف النسبة بين تكلفة الخردة وتكلفة إعادة التشغيل.

تم اقتراح نموذجين رياضيين لتحديد القيمة المتوسطه المثلى لعملية الإنتاج في حالة تعظيم الفائدة المتوقعة من عملية الإنتاج وفى حالة تقليل التكلفة المتوقعة من عملية الإنتاج.

تم عرض مثال لتوضيح كيفية استخدام الأسلوب المقترح في هذه الدراسة وتحديد مدى حساسية القيمة المتوسطه المثلى لعملية الإنتاج عند اختلاف النسبة بين تكلفة الخردة وتكلفة إعادة التشغيل.وقد تم التوصل إلى النتائج التالية:-

1- طبقا لنوع التوزيع الذي تتبعه صفة الجودة يمكن تحديد القيمة المتوسطه المثلى لعملية الإنتاج وذلك باستخدام النموذج الرياضي المقترح في هذه الدراسة.
2- في حالة إتباع صفة الجودة التوزيع الطبيعي أو التوزيع الغير طبيعي تكون القيمة المتوسطه المثلى لعملية الإنتاج باستخدام نموذج تعظيم الفائدة اكبر من القيمة المثلى باستخدام نموذج تقليل التكلفة المتوقعة لعملية الإنتاج.