DETERMINATION OF ALL STABILIZING PI CONTROLLERS FOR LINEAR TIME INVARIANT SYSTEMS

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Received 15 June 2013; accepted 1 August 2013

ABSTRACT

This paper considers the problem of controlling a given linear time invariant (LTI) system described by a rational transfer function using a PI controller. A method based on graphical and computational tools has been proposed. This method specifies a PI controller $(K_p \text{ and } K_i)$ that satisfies both frequency domain specifications (a specified gain or phase margin) and minimum integral time absolute error (ITAE).

Keywords: ITAE, LTI, Optimal gain and phase margins, PI controller, Robustness, Stabilizing region.

1. Introduction

There have been many research works on the tuning of PI and PID controllers because these controllers have simple structures and have been used in industries. [1-2].

Consider a real polynomial $\delta(s)$ of degree n, $\delta(s)$ is Hurwitz if all roots of $\delta(s)=0$ lie in the open left half s-plane (Re[s] < 0). The problem of checking the stability of the polynomial $\delta(s)$ can be solved using classical criteria such as the Nyquist stability criterion and the Routh-Huriwtz criterion, it is not easy to extend these methods to the cases where PI or PID controllers are involved.

In this paper a simple method in the frequency domain is proposed to tune a PI controller for a LTI system. This method gives another way to determine the stability boundary locus which was presented in [3-6].

In [3-4], the authors, equated the real and the imaginary parts of the characteristic equation in the frequency domain to zero, then they solved both equations to find K_p and K_i in terms of the frequency variable. They plotted the obtained values of (K_p, K_i) in a plane. The determination of the stability boundary was done by choosing a test point within each region.

In [7], the authors gave a computational method for the stabilizing range of K_p and K_i. They used a set of strings that are generated to satisfy specified requirements using the generalized Hermite-Biehler theorem.

In [8], the generalized Mikhailov criterion showed that, the number of roots on the imaginary axis of the characteristic polynomial $\delta(s)$ equals the number of the common zeros of the real part and the imaginary part of the characteristic polynomial $\delta(s)$. The

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system can be marginally stable or unstable depending on the multiplicity of these imaginary roots taking into consideration that no roots are in the RHP.

In [9], the generalized Mikhailov criterion is utilized to give a solution to the problem of finding the stabilizing gains of PI controller for a given LTI plant.

In both [7] and [9], the gains K_p and K_i are separated to find the stabilizing range. The ranges of K_p that satisfy the condition for the existence of a stabilizing K_i value at each fixed value of K_p was determined by applying the root locus ideas.

In this paper, we propose a simple and fast method to find stability boundary of all stabilizing PI controllers and also those that achieve specified gain and phase margins for a given delay-free LTI plant. This method does not need to apply the root locus ideas as in [7, 9] or generate a set of strings as in [7]. The method depends on equating the real and the imaginary parts of the modified characteristic equation in the frequency domain to zero and finding two separate equations for K_p and K_i in terms of the frequency variable directly without solving equations as in [3-6]. We then compute both stabilizing gains by sweeping over the frequency values algebraically without using a test point as in [3-6].

Gain and phase margins have been used as measures of robustness. A classical approach that takes model uncertainties into consideration is to design the closed-loop control system with sufficient gain and phase margins. We use our proposed method to find all stabilizing PI controllers that achieve a specified gain margin or phase margin.

After determining this stability boundary, we can search for the stabilizing PI controller that gives minimum integral time absolute error (ITAE) for a given plant at the specified gain margin or phase margin. The resulting PI controller will satisfy both robustness and performance requirements.

The specified gain and phase margin in this case are considered as optimal gain and phase margins.

2. Calculation of all stabilizing PI controllers

Consider the feedback control system shown in Fig. 1, where

$$G(s) = \frac{N(s)}{D(s)} \tag{1}$$

with $N(s), D(s)$ being coprime polynomials and degree $[N(s)] \leq$ degree $[D(s)]$.

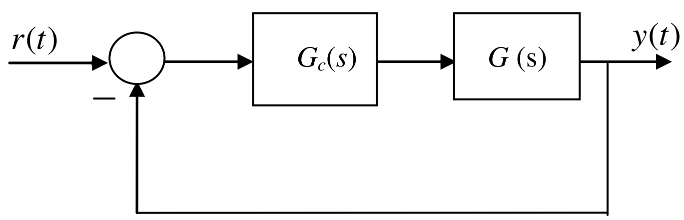

Fig. 1. Feedback control system

The transfer function of PI controller is:

$$G_c(s) = \frac{K_p s + K_i}{s} \tag{2}$$

Then the closed-loop characteristic polynomial $\delta(s)$ is:

$$\delta(s) = s\,D(s) + (K_i + K_p s)N(s) = 0 \tag{3}$$

Consider the even-odd decompositions of $N(s)$ and $D(s)$:

$$N(s) = N_e(s^2) + sN_o(s^2) \text{ , and}$$
$$D(s) = D_e(s^2) + sD_o(s^2)\,. \tag{4}$$

so that

$$\delta(s) = [s^2 D_o(s^2) + s^2 K_p N_o(s^2) + K_i\, N_e(s^2)] + s[D_e(s^2) + K_p N_e(s^2) + K_i\, N_o(s^2)] = 0 \tag{5}$$

From (5), both the even and the odd parts of $\delta(s)$ depend on K_p and K_i which causes difficulties when trying to find the ranges of K_p and K_i that make the system stable. To overcome this problem, we construct a polynomial where the even part of this polynomial depends on K_i and the odd part depends on K_p [7].

Define:

$$N^*(s) = N(-s) = N_e(s^2) - sN_o(s^2) \tag{6}$$

then

$$\delta(s)N^*(s) = [s^2 X_1(s^2) + K_i X_2(s^2)] + s[X_3(s^2) + K_p X_2(s^2)] = 0 \tag{7}$$

where

$$X_1(s^2) = N_e(s^2)D_o(s^2) - D_e(s^2)N_o(s^2), \tag{8}$$
$$X_2(s^2) = N_e(s^2)N_e(s^2) - s^2 N_o(s^2)N_o(s^2), \tag{9}$$
$$X_3(s^2) = D_e(s^2)N_e(s^2) - s^2 D_o(s^2)N_o(s^2). \tag{10}$$

Substituting s = jw, we obtain

$$\delta(jw)N^*(jw) = \big(R(w) + jI(w)\big)\big(N_R^*(w) + jN_I^*(w)\big) = U(w, K_i) + jV(w, K_p) = 0 \tag{11}$$

where $R(w)$ and $N_R^*(w)$ are real parts of $\delta(jw)$ and $N^*(jw)$ respectively. $I(w)$ and $N_I^*(w)$ are imaginary parts of $\delta(jw)$ and $N^*(jw)$ respectively, and

$$N_R^*(w) = N_e(-w^2) \text{ ,} \tag{12}$$

$$N_I^*(w) = -wN_o(-w^2), \qquad (13)$$

$$U(w, K_i) = U_1(w) + K_i U_2(w), \qquad (14)$$

$$V(w, K_p) = V_1(w) + K_p V_2(w), \qquad (15)$$

$$U_1(w) = -w^2(D_o(-w^2)N_e(-w^2) - D_e(-w^2)N_o(-w^2)), \qquad (16)$$

$$U_2(w) = N_e(-w^2)N_e(-w^2) + w^2 N_o(-w^2)N_o(-w^2), \qquad (17)$$

$$V_1(w) = w(D_e(-w^2)N_e(-w^2) + w^2 D_o(-w^2)N_o(-w^2)). \qquad (18)$$

and

$$V_2(w) = w(N_e(-w^2)N_e(-w^2) + w^2 N_o(-w^2)N_o(-w^2)). \qquad (19)$$

From these expressions, we note that K_i and K_p appear in $U(w, K_i)$ and $V(w, K_p)$ respectively.
From (11), we obtain:

$$U(w, K_i) = 0 \ \ and \ \ V(w, K_p) = 0 \qquad (20)$$

So (20) can be rewritten as,

$$K_p = -\frac{V_1(w)}{V_2(w)} \qquad (21)$$

$$K_i = -\frac{U_1(w)}{U_2(w)} \qquad (22)$$

Hence, the stabilizing region of (K_p, K_i) values can be plotted with w varies from a zero to a finite value $w = w_0$. The frequency w_0 is frequency value of a significant point on the Nyquist plot of the plant where $\angle G(jw_0) = \pm 180^o$ or $\angle G(jw_0) = 0^o$ (Imaginary part of $G(jw) = 0$).

2.1. The steps of the proposed method

1- Initialize $w = 0$, step = 0.005 and $w_{max} = w_0$

2- Compute K_p and K_i using (21) and (22), then check the stability of the closed-loop system at this point (K_p, K_i) by finding the roots of $\delta(s)$.

3- If the system is stable, find the other positive real value of $w \le w_{max}$ at the same K_p obtained in step (2) by solving the equation $V(w, K_p) = 0$.

4- At this value of w obtained from step (3), compute K_i from (22). Check the stability of the closed-loop system. The value of K_i where the system is stable will be the second value at the fixed value of K_p obtained before in step (2). Since the stabilizing region is two dimensional space (2D), so for every value of K_p, there are two values of K_i, the lower bound $\boldsymbol{K_{i\,\min}}$ and the upper bound $\boldsymbol{K_{i\,\max}}$.

5- Increase w as follows: $w = w$ + step

6- If $w \le w_{\max}$ go to step (2), else stop the method.

We write a MATLAB program to run this algorithm.

3. All stabilizing PI controllers for specified gain and phase margins

In this section, we consider the problem of finding all stabilizing PI controllers that achieve specified gain and phase margins for a given LTI plant. Let $2 \le A_m \le 5$ and $30^o \le \theta_m \le 60^o$ denote the desired gain and phase margin [10].
Consider Fig. 2 with a gain-phase margin tester [11],

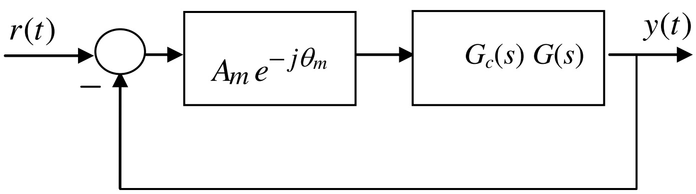

Fig. 2. Feedback control system with gain-phase tester.

Then the characteristic equation can be written as follows:

$$\delta(s) = s\,D(s) + A_m\,e^{-j\theta_m}\,(K_i + K_p s)N(s) = 0 \tag{23}$$

Multiply Eq.(23) by $e^{j\theta_m}$, we obtain

$$\delta(s) = e^{j\theta_m} s\,D(s) + A_m\,(K_i + K_p s)N(s) = 0 \tag{24}$$

(9) will be modified to the new equation as:

$$\delta(jw)N^*(jw) = U(w,K_i) + jV(w,K_p) = 0 \tag{25}$$

where $U(w,K_i)$ and $V(w,K_p)$ are defined in (14) and (15) with new equations of $U_1(w)$, $U_2(w)$, $V_1(w)$, and $V_2(w)$ are as follows:

$$U_1(w) = -w^2\cos(\theta_m)(D_o(-w^2)N_e(-w^2) - D_e(-w^2)N_o(-w^2)) - w\sin(\theta_m)$$
$$+ w^2 D_o(-w^2)N_o(-w^2)), \tag{26}$$

$$U_2(w) = A_m(N_e(-w^2)N_e(-w^2) + w^2 N_o(-w^2)N_o(-w^2)), \tag{27}$$

$$V_1(w) = -w^2 \sin(\theta_m)(D_o(-w^2)N_e(-w^2) - D_e(-w^2)N_o(-w^2)) + w\cos(\theta_m)$$
$$(D_e(-w^2)N_e(-w^2) + w^2 D_o(-w^2)N_o(-w^2)), \quad (28)$$

and

$$V_2(w) = w A_m (N_e(-w^2)N_e(-w^2) + w^2 N_o(-w^2)N_o(-w^2)). \quad (29)$$

The values of K_p and K_i on the stability boundary can be computed using the (21) and (22).

We have two design specifications to obtain the stability boundary of (K_p, K_i) as follows:

For design specification on a specified gain margin A_m, we set $\theta_m = 0$ in (26) and (28).

For design specification on a specified phase margin θ_m, we set $A_m = 1$ in (27) and (29).

In both cases, the steps proposed in section 2.1 can now be used to solve this problem. But for the case of the phase margin, the frequency w_0 satisfies the relation $\angle G(jw_0) = -180^o + \theta_m$ is considered. From this stability boundary, we can search for the stabilizing PI controller that gives minimum ITAE for a given plant, so the optimal gain or phase margin can be achieved.

The ITAE is defined as:

$$ITAE = \int_0^\infty t|e(t)|dt \quad (30)$$

This optimization is made with the response obtained via simulation using MATLAB program. We compute the unit step response of the closed-loop system for all stabilizing PI controllers that lie on the stability boundary that satisfies the specified A_m or θ_m, depending on the design specification. Then we compute the error (e(t) = y(t) – r(t)) and applying the trapezoidal rule to approximate the integration in (30). Finally we search for stabilizing point (K_p, K_i) that gives minimum value of ITAE.

We choose ITAE since the corresponding response of this criterion tends to have smaller settling time and overshoot than that of the integral squared error (ISE) and the integral absolute error (IAE) because of the time weighting factor in (30).

4. Numerical Example

Consider the control system in Fig. 1 with the plant [7],

$$G(s) = \frac{N(s)}{D(s)} = \frac{s^4 + 6s^3 + 12s^2 + 54s + 16}{s^5 + 11s^4 + 22s^3 + 60s^2 + 47s + 25}$$

The first aim is to find all stabilizing PI controllers that make the characteristic polynomial of (3) Hurwitz.

From (16), (17), (18), and (19)

$$U_1(w) = -w^{10} - 32w^8 + 627w^6 - 2474w^4 + 598w^2,$$
$$U_2(w) = w^8 + 12w^6 - 472w^4 + 2532w^2 + 256,$$
$$V_1(w) = 5w^9 - 6w^7 - 549w^5 + 1278w^3 + 400w,$$
$$V_2(w) = w^9 + 12w^7 - 472w^5 + 2532w^3 + 256w.$$

So (21) and (22) will be as follows:

$$K_p = \frac{-5w^9 + 6w^7 + 549w^5 - 1278w^3 - 400w}{w^9 + 12w^7 - 472w^5 + 2532w^3 + 256w},$$

$$K_i = \frac{w^{10} + 32w^8 - 627w^6 + 2474w^4 - 598w^2}{w^8 + 12w^6 - 472w^4 + 2532w^2 + 256},$$

By following the steps presented in section 2.1, where $w \in [0, w_0]$ and $w_0 = 2.4162$ rad/sec, we can determine the stabilizing region (K_p, K_i) as shown in Fig. 3, (The outer curve).

The range of K_p for stable system is:

$$-0.787425 < K_p < 2.484578$$

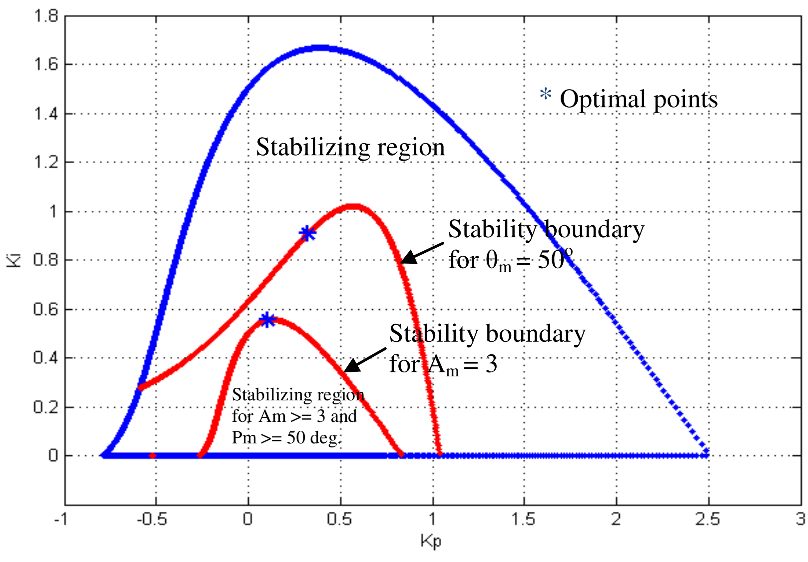

Fig. 3. The stabilizing regions

The second aim is to find all stabilizing PI controllers which satisfy the conditions that the phase margin of the system is greater than 50^o or the gain margin is greater than 3 (9.54 db). So we compute the stability boundary using (26), (27), (28), and (29).

1. For $A_m = 3$ and $\theta_m = 0$,
From (26), (27), (28), and (29)

$$U_1(w) = -w^{10} - 32w^8 + 627w^6 - 2474w^4 + 598w^2,$$
$$U_2(w) = 3w^8 + 36w^6 - 1416w^4 + 7596w^2 + 768,$$
$$V_1(w) = 5w^9 - 6w^7 - 549w^5 + 1278w^3 + 400w,$$
$$V_2(w) = 3w^9 + 36w^7 - 1416w^5 + 7596w^3 + 768w.$$

So (21) and (22) will be as follows:

$$K_p = \frac{-5w^9 + 6w^7 + 549w^5 - 1278w^3 - 400w}{3w^9 + 36w^7 - 1416w^5 + 7596w^3 + 768w},$$
$$K_i = \frac{w^{10} + 32w^8 - 627w^6 + 2474w^4 - 598w^2}{3w^8 + 36w^6 - 1416w^4 + 7596w^2 + 768},$$

By following the steps presented in section 2.1, where $w \in [0, 2.4162]$ and searching only for $K_{i\max}$, we can draw the stability boundary (K_p, K_i) as shown in Fig. 3.
The range of K_p for stable system is:

$$-0.262475 < K_p < 0.828193$$

After determining this stability boundary, we can search through all the points forming this boundary to find the (K_p, K_i) point that gives minimum ITAE. This point is represented by a blue notation * in Fig. 3.
The resulting PI controller transfer function is:

$$G_c(s) = \frac{K_p s + K_i}{s} = \frac{0.106633s + 0.554035}{s}$$

The bode plot of the open-loop transfer function is shown in Fig. 4. In this case, A_m is called optimal gain margin.
The step response of the closed-loop system with minimum ITAE at $A_m = 3$ is shown in Fig. 5.

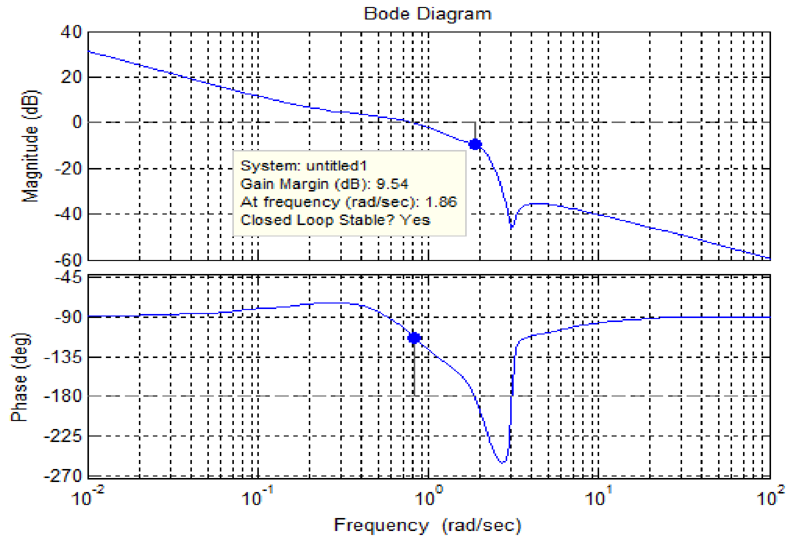

Fig. 4. The bode plot of the open-loop transfer function at optimal $A_m = 3$.

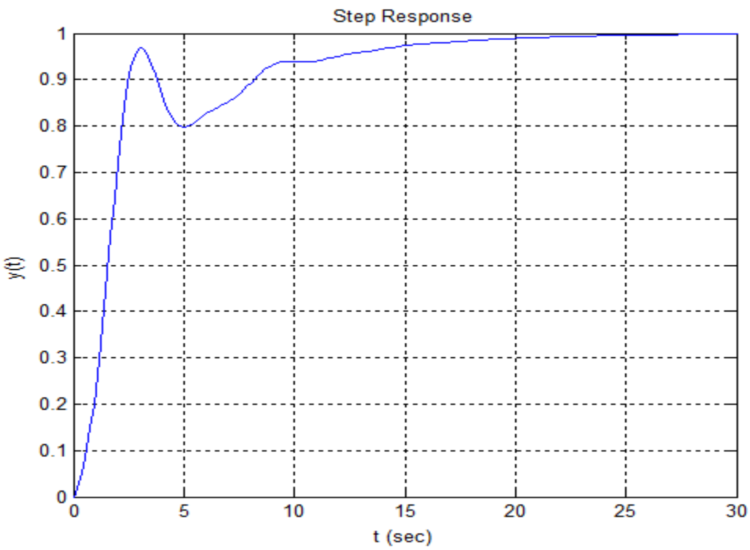

Fig. 5. The step response of the closed-loop system with minimum ITAE at $A_m = 3$.

2. For $A_m = 1$ and $\theta_m = 50^o$,

From (26), (27), (28), and (29)

$$U_1(w) = -0.64279w^{10} - 3.8302w^9 - 20.5692w^8 + 4.5963w^7 + 403.0278w^6 + 420.5584w^5 - 1590.2565w^4 - 979.0048w^3 - 384.387w^2 - 306.4178w,$$

$$U_2(w) = w^8 + 12w^6 - 472w^4 + 2532w^2 + 256,$$

$$V_1(w) = -0.76604w^{10} + 3.2139w^9 - 24.5134w^8 - 3.8567w^7 + 480.3099w^6 - 352.8904w^5 - 1895.194w^4 + 821.4826w^3 + 458.0946w^2 + 257.115w,$$

$$V_2(w) = w^9 + 12w^7 - 472w^5 + 2532w^3 + 256w.$$

So (21) and (22) will be as follows:

$$K_p = \frac{0.76604w^{10} - 3.2139w^9 + 24.5134w^8 + 3.8567w^7 - 480.3099w^6 + 352.8904w^5 + 1895.194w^4 - 821.4826w^3 - 458.0946w^2 - 257.115w}{w^9 + 12w^7 - 472w^5 + 2532w^3 + 256w},$$

$$K_i = \frac{0.64279w^{10} + 3.8302w^9 + 20.5692w^8 - 4.5963w^7 - 403.0278w^6 - 420.5584w^5 + 1590.2565w^4 + 979.0048w^3 + 384.387w^2 + 306.4178w}{w^8 + 12w^6 - 472w^4 + 2532w^2 + 256}$$

By following the steps presented in section 2.1, and searching only for $K_{i\max}$, we can draw the stability boundary (K_p, K_i) as shown in Fig. 3. The range of w needed for stabilization for this case can be found from $\angle G(jw_0) = -130^o$, so the range will be $w \in [0, 2.0201]$.
The range of K_p for stable system is:

$$-0.586993 < K_p < 1.038473$$

After determining this stability boundary, we can search through all the points forming this boundary to find the (K_p, K_i) point that gives minimum ITAE. This point is represented by a blue notation * in Fig. 3.
The resulting PI controller transfer function is:

$$G_c(s) = \frac{K_p s + K_i}{s} = \frac{0.324399s + 0.907103}{s}$$

The bode plot of the open-loop transfer function is shown in Fig. 6. In this case, θ_m is called optimal phase margin. The step response of the closed-loop system with minimum ITAE at $\theta_m = 50^o$ is shown in Fig. 7.

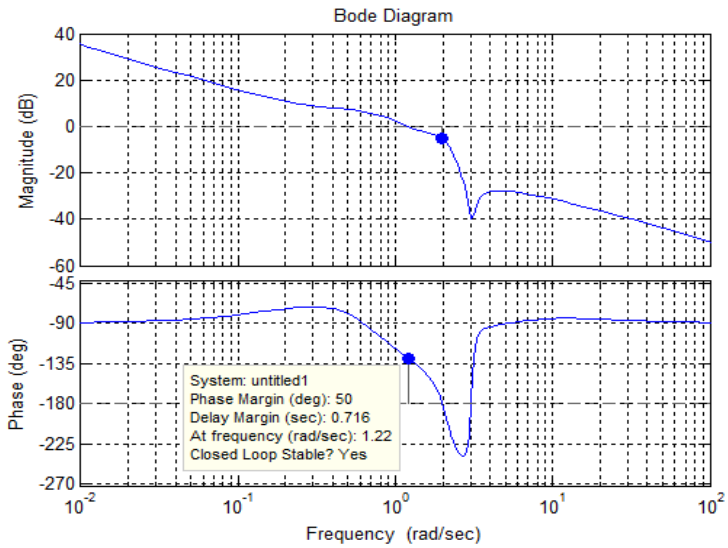

Fig. 6. The bode plot of the open-loop transfer function at optimal $\theta_m = 50^o$.

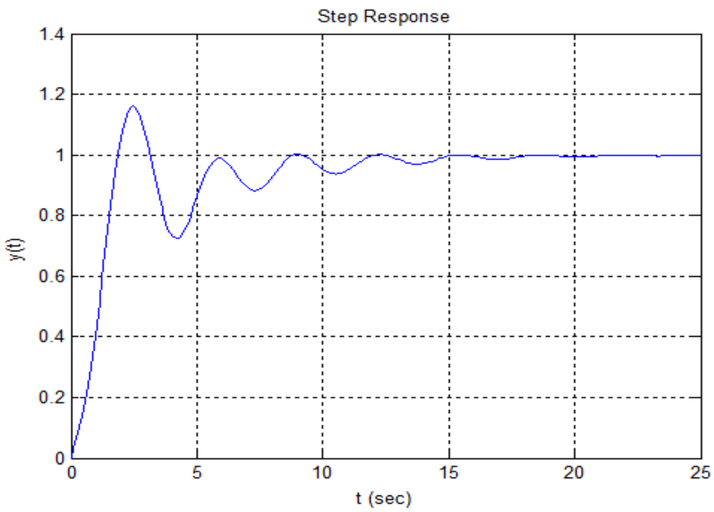

Fig. 7. The step response of the closed-loop system with minimum ITAE at $\theta_m = 50^o$.

5. Conclusions

In this paper, we proposed a simple method to compute all stabilizing PI gains limits. This method depends on equating the real and the imaginary parts of the modified characteristic equation to zero and finding two separate equations for K_p and K_i. We then compute both gains by sweeping over the frequency values.

By this method, we can also find all stabilizing PI controllers that achieve specified gain and phase margins. From the stability boundary, we find easily the gains of the PI controller that gives optimal gain or phase margin by searching for minimum ITAE. Our future work is to investigate PI/PID controllers with time-delay systems.

6. References

[1] K. J. Astrom and T. Hagglund, PID Controllers: Theoy, Design, and Tuning. Instrument Society of America, 1995.

[2] W. K. Ho, C. C. Hang, and L. S. Cao, "Tuning of PID controllers based on gain and phase margins specifications," Automatica, vol. 31, pp. 497-502, 1995.

[3] N. Tan, and I. Kaya, "Computation of stabilizing PI and PID controllers," Proceedings of 2003 IEEE Conference on Control Applications, CCA, vol.2, pp. 876-881, 2003.

[4] N. Tan, I. Kaya, C. Yeroglu and D. P. Atherton "Computation of stabilizing PI and PID controllers using the stability boundary locus," Energy Conversion and Management, vol.47, issues 18-19, pp. 3045-3058, 2006.

[5] N. Tan, "Computation of stabilizing PI-PD controllers," International Journal of Control, Automation, and Systems, vol. 7, no. 2, pp.175-184, 2009.

[6] R. Matusu, R. Prokop, and Z. Prokopova, "Simple Tuning of PI Controllers for Interval Plants," Manuf. and Ind. Eng., © Faculty of Manuf. Tech. TUKE, issue 11, vol. 2, pp. 44-47, 2012.

[7] G. J. Silva, A. Datta, and S. P. Bhattacharyya, "PID Controllers for Time-Delay Systems," Birkhäuser, Boston, 2005.

[8] A. I. Saleh, M. M. Hasan, and N. M. M. Darwish, "The Mikhailov Stability Criterion Revisited," Journal of Engineering Sciences, Published by Faculty of Engineering-Assiut University-Egypt, vol. 38, no. 1, pp. 195-207, January 2010.

[9] A. I. Saleh, M. M. Hasan, and N. M. M. Darwish, "Determination of Controllers Gains Limit Using The Mikhailov Stability Criterion," Journal of Engineering Sciences, Published by Faculty of Engineering-Assiut University-Egypt, vol. 38, no. 1, pp. 209-219, January 2010.

[10] W. K. HO, K. W. LIM, and W. XU, "Optimal Gain and Phase Margin Tuning for PID Controllers," Automatica, vol. 34, no. 8, pp. 1009-1014, 1998.

[11] C. H. Chang, and K. W. Han, "Gain margins and phase margins for control systems with adjustable parameters," Journal of Guidance, Control, and Dynamics, vol. 13, no. 3, pp. 404-408, 1990.

تحديد كل المحكمات التناسبية التكاملية التى تضمن الاتزان للأنظمة الخطية الثابتة مع الزمن

الملخص العربى

هذا البحث يتناول كيفية التحكم فى أى نظام خطى ثابت مع الزمن والذى يتم وصفه بدالة تحويل فى مجال التردد باستخدام محكم تناسبى تكاملى. حيث تم اقتراح طريقة لايجاد كل المحكمات التناسبية التكاملية التى تضمن اتزان نظام التحكم وذلك حسابيا من خلال المعادلات المستنتجة ورسوميا عن طريق رسم العلاقة بين ثوابت التناسب وثوابت التكامل لتلك المحكمات.

من خلال هذه الطريقة، أوجدنا أيضا المحكم التناسبى التكاملى (ثابتى التناسب والتكامل) والذى يحقق هامش كسب وهامش طور محددين مسبقا وذلك فى مجال التردد وفى نفس الوقت يعطى أقل قيمة لتكامل حاصل ضرب مقياس الخطأ والزمن.

ولقد أعطي مثال توضيحي يبين الطريقة المقترحة لايجاد معاملات المحكمات التناسبية التكاملية والتى تضمن الاتزان ومن ثم ايجاد المحكم الذى يحقق المواصفات المطلوبة فى مجالى التردد والزمن.